Transformation of Differential – Algebric Equations system to a system of differential equations using index reduction

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Abstract:

In this paper, the system of differential – algebriaic equations has been transformed to a system of differential equations using index reduction , then the transformed systems was tested for stability by Sylvester's criterion depending on Liapunov functions .

Keywords : Sylvester's criterion , index reducation , semi – explicit differential algebraic equations .

1 Introduction

Marz[1] , after describing differential algebraic equation as " Singular ODE ," introduces them as " special implicit ordinary differential equations (ODE's) $f(\dot{x},t),x(t),t)=0$ where the partial Jacobian fy (y , x , t) is singular for all values of its arguments " .

Differential Algebraic Equations arise naturally in many applications and have been known by a variety of names depending on the area of application.

They may be known , for example , as descriptor system (circuit analysis) , generalized state space (system theory) , constrained systems , reduced order model and non – standard systems . In the simulation of physical problems the model often takes the form of DAE , consisting of collection of relationships between the variables involved and some of their derivatives .

2 – Construction of Liapunov Functions for Differential - Algebraic Equations

In this section , we will introduce a new modification for Sylvester's criterion for constructing Liapunov functions of Differential - Algebraic Equations .

2.1 Sylvester's Criterion [2]

Let the definite function V = V(x) as well as its derivatives be continuous functions, then at $X_1 = X_2 = \ldots = X_n = 0$, it has an isolated extremum and hence all the partial derivatives of the first order calculated at this point are equal to zero (necessary conditions for the existence of an extremum) .

$$\left(\frac{\partial v}{\partial \mathrm{x_j}}\right)_{\mathrm{xj=0}} = 0 \qquad j = 1\,,2\,,\ldots,n$$

Expanding the function V by a Maclaurian series in powers of X_1, X_2 , X_3, ... X_n as follows :

$$V = V(0) + \sum_{j=1}^{n}\left(\frac{\partial V}{\partial_{xj}}\right)_{\mathrm{xj=0}} \mathrm{xj} + \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial^2 \mathrm{V}}{\partial\, \mathrm{x_k}\ \partial\, \mathrm{x_j}}\right)_{xj=0} x_k\ x_j + \ldots,$$

Account relation V(0) = 0 and

$$(\frac{\partial v}{\partial x_j})_{xj=0} = 0 \qquad j = 1\,,2\,,\ldots,n$$

We get $V(x) = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} C_{kj}X_kX_j + \cdots,$

Here the constant $C_{kj} = C_{jk}$ are defined as :

$$C_{kj} = (\frac{\partial^2 V}{\partial X_k\ \partial X_{xj}})_{xj} = 0 \quad (j = 1\,,2,\ldots,n)$$

From

$$V(x) = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} C_{kj}X_kX_j + \ \ldots,$$

It follows the expansion of the definite function V into power series of X_1, X_2 , X_3, ... X_n does not contain any linear terms.

Assuming that the quadratic form

$$\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} C_{kj}X_kX_j$$

It always positive definite and vanishes only for X_1, $X_2 = X_3 = \ldots = X_n$

Let the matrix of coefficients of the quadratic from

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ & & \vdots & & \\ . & . & \ddots & & . \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{bmatrix}$$

And write its n principal diagonal minors in the matrix above .

$$\Delta 1 = C_{11} > 0$$

$$\Delta 2 = \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} > 0$$

$$\Delta n = \begin{vmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{vmatrix} > 0$$

It is necessary and sufficient that all principal diagonal minors $\Delta_1, \Delta_2, \ldots \Delta_n$ of it cofficients matrix be positive.

If the function V is negative definite then the function V is positive definite and the sufficient condition for negative definiteness of function V is the Sylvester's criterion for the matrix - C ,

This criterion has the form

$\Delta_1 < 0 \qquad \Delta_2 > 0, \ldots$

i .e The determinants Δj should alternately change their signs , and the sign of $\Delta 1 = C_{11}$ should be negative .

Example [2]

Consider the function .

$V(x) = 1 + sin^2 X_1 - \cos(X_1 - X_2)$

$sin^2 X_1 = x_1^2 + \ldots., \cos(X_1 - X_2) = 1 - {}^1/_2 (X_1 - X_2)^2 + \ldots,$

Substituting the expressions for $sin^2 X_1$ and

$\cos(X_1 - X_2)$ into the function V , we get

$V(x) = 1 + X_1{}^2 - 1 + {}^1/_2 (X_1 - X_2)^2 + \cdots,.$

After some simplification

$V(x) = \frac{1}{2}(3X_1{}^2 - 2X_1X_2 + X_2{}^2 + \ldots$

The elements C_{12} and C_{21} are equal to one half of the coefficient of the $X_1 X_2$ term: $\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$

Now , if we calculate the principal diagonal minors :

$\Delta 1 = 3 \qquad \Delta 2 = \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 2$

Then Sylvester's criterion is satisfied i . e

all $\Delta j > 0$, and therefore the function V considered here is positive definite.

2 . 2 Reduction [3]

Reduction is the most suitable to deal with some formulas of Differential Algebraic Equations system like quasilinear and Hesnborg [4]

The best way to reduce

$X^{\circ} = f(X, y)$

$0 = g(X)$

Is to differentiate its constraint then evaluate in the differential part , this reduction is called index reducation [5] .Differentiate

$g(X) = 0$

Implicitly yield :

$g_x x^{\circ} = 0 \rightarrow g_x f(x,y) = 0$

Define

$\tilde{g}(x,y) = g_x f(x,y)$

We get

$X^{\circ} = f(x,y)$

$0 = \tilde{g}(x,y)$

Differentiation of the second part given

$\tilde{g}_x x^{\circ} + \tilde{g}_y y^{\circ} = 0$

Which can be reduced if $\tilde{g}_y$ is non singular ($\tilde{g}_y^{-1}$exist).

Since

$\tilde{g}_y = \tilde{g}_x f_y$

Which is non – singular then we get the explicit system

$y^{\circ} = -\tilde{g}_y^{-1} \tilde{g}_x \ f(x,y)$.

Now , we introduce the new approach for construction of Liapunov function of Differential Algebraic Equations .

Approach (2.1.1)

This approach is based on the Sylvester's Criterion

Consider the system of differential Algebraic Equation

$X^{\circ} = f(x,y)$

$0 = g(x, y)$

Differentiation of second part given

$0 = g_x x^{\cdot} + g_y y^{\cdot}$

$y^{\cdot} = -g_y^{-1}(x, y) g_x(x, y) y^{\cdot}$

When g_y^{-1} exist (i , e det . $(g_y) \neq 0$) , we have

$X^{\cdot} = f(x, y)$

$y^{\cdot} = -g_y^{-1}(x, y) g_x(x, y) x^{\cdot}$

And the differentiation index one

This system non singular

Ordinary differential Equations

Example 2.1.2

Consider system of differential Algebraic Equations

$x_1^{\cdot} = x_1^2$

$0 = 2x_1 - x_2 \rightarrow 2x_1 = x_2$

$x_1^{\cdot} = x_1^2$

$x_2^{\cdot} = 2x_1^{\cdot} \quad \rightarrow \quad x_2^{\cdot} = 2(x_1^2)$

$V(x) = \frac{1}{2}(6x_1^2)$

$\Delta 1 = 6 > 0$

Then the Sylvester's Criterion is satisfied , i , e $\Delta 1 > 0$ and therefore the function considered positive definite.

Example 2.1.3

$\overset{\circ}{x}_1 = x_1^2 + x_2^2$

$0 = x_1 - x_2 \rightarrow x_1 = x_2 \rightarrow \dot{x}_1 = \dot{x}_2$

$\overset{\circ}{x}_2 = x_1^2 + x_2^2$

$V(x) = \frac{1}{2}(4\,x_1^2 + 4\,x_2^2)$

$C_{11} = \Delta_1 = 4 > 0$

$\Delta_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 16 > 0$

Then , Sylvester's Criterion is satisfied

i , e All $\Delta_j > 0$ j = 1 , 2 and therefore the function considered positive definite .

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تحويل نظام المعادلات الجبرية التفاضلية الى نظام معادلات تفاضلية اعتيادية بأستخدام طريقة التقليص

الخلاصة :

في هذا البحث ، قمنا بتحويل المعادلة التفاضلية الجبرية الى معادلة تفاضلية اعتيادية بأستخدام طريقة التقليص ثم تحويل منظومة الحل لاختبار الاستقرارية بأستخدام صيغة سلفستر على دوال ليبانوف